

Ex 4

☞ Let R be a relation from a set A to a set B . The **inverse relation** from B to A , denoted by R^{-1} , is the set of ordered pairs $\{(b, a) \mid (a, b) \in R\}$. The **complementary relation** \overline{R} is the set of ordered pairs $\{(a, b) \mid (a, b) \notin R\}$.

Let R be the relation $R = \{(a, b) \mid a < b\}$ on the set of integers. Find

a) R^{-1} .

b) \overline{R} .

DEFINITIONS

The **inverse relation** R^{-1} is the set $\{(b, a) | (a, b) \in R\}$

The **complementary relation** \bar{R} is the set $\{(a, b) | (a, b) \notin R\}$

SOLUTION

$$R = \{(a, b) | a < b\}$$

(a) Use the definition of the inverse relation:

$$R^{-1} = \{(b, a) | (a, b) \in R\}$$

$$= \{(b, a) | a < b\}$$

$$= \{(a, b) | b < a\}$$

$$= \{(a, b) | a > b\}$$

Interchange a and b

(b) Use the definition of the complementary relation:

$$\begin{aligned}\overline{R} &= \{(a, b) \mid (a, b) \notin R\} \\ &= \{(a, b) \mid a \not\prec b\} \\ &= \{(a, b) \mid a \geq b\}\end{aligned}$$

Ex 5

$R_1 = \{(a, b) \in \mathbf{R}^2 \mid a > b\}$, the “greater than” relation,

$R_2 = \{(a, b) \in \mathbf{R}^2 \mid a \geq b\}$, the “greater than or equal to” relation,

$R_3 = \{(a, b) \in \mathbf{R}^2 \mid a < b\}$, the “less than” relation,

$R_4 = \{(a, b) \in \mathbf{R}^2 \mid a \leq b\}$, the “less than or equal to” relation,

$R_5 = \{(a, b) \in \mathbf{R}^2 \mid a = b\}$, the “equal to” relation,

$R_6 = \{(a, b) \in \mathbf{R}^2 \mid a \neq b\}$, the “unequal to” relation.

Find

a) $R_1 \cup R_3$.

b) $R_1 \cup R_5$.

c) $R_2 \cap R_4$.

d) $R_3 \cap R_5$.

e) $R_1 - R_2$.

f) $R_2 - R_1$.

g) $R_1 \oplus R_3$.

h) $R_2 \oplus R_4$.

Find

Find

a) $R_1 \circ R_1$.

b) $R_1 \circ R_2$.

c) $R_1 \circ R_3$.

d) $R_1 \circ R_4$.

e) $R_1 \circ R_5$.

f) $R_1 \circ R_6$.

g) $R_2 \circ R_3$.

h) $R_3 \circ R_3$.

DEFINITIONS

Union $A \cup B$: All elements that are either in A OR in B

Intersection $A \cap B$: All elements that are both in A AND in B .

Difference $A - B$: All elements in A that are NOT in B (complement of B with respect to A).

Symmetric difference $A \oplus B$: All elements in A or in B , but not in both.

SOLUTION

$$R_1 = \{(a, b) \in \mathbf{R}^2 \mid a > b\}$$

$$R_2 = \{(a, b) \in \mathbf{R}^2 \mid a \geq b\}$$

$$R_3 = \{(a, b) \in \mathbf{R}^2 \mid a < b\}$$

$$R_4 = \{(a, b) \in \mathbf{R}^2 \mid a \leq b\}$$

$$R_5 = \{(a, b) \in \mathbf{R}^2 \mid a = b\}$$

$$R_6 = \{(a, b) \in \mathbf{R}^2 \mid a \neq b\}$$

(a) The union of two relations contains all ordered pairs that are in either relation.

$$\begin{aligned} R_1 \cup R_3 &= \{(a, b) \in \mathbf{R}^2 \mid a > b \text{ or } a < b\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a \neq b\} \\ &= R_6 \end{aligned}$$

Because when $a < b$ or $b > a$, then we know that a and b cannot be the same (although we do not know anything more).

(b) The union of two relations contains all ordered pairs that are in either relation.

$$\begin{aligned}R_1 \cup R_5 &= \{(a, b) \in \mathbf{R}^2 \mid a > b \text{ or } a = b\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a \geq b\} \\ &= R_2\end{aligned}$$

(c) The intersection of two relations contains all ordered pairs that are in both relations.

$$\begin{aligned}R_2 \cap R_4 &= \{(a, b) \in \mathbf{R}^2 \mid a \geq b \text{ and } a \leq b\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a = b\} \\ &= R_5\end{aligned}$$

(d) The intersection of two relations contains all ordered pairs that are in both relations.

$$\begin{aligned}R_3 \cap R_5 &= \{(a, b) \in \mathbf{R}^2 \mid a < b \text{ and } a = b\} \\ &= \emptyset\end{aligned}$$

Since $a < b$ and $a = b$ can never be true at the same time, $R_3 \cap R_5$ cannot contain any ordered pairs.

(e) $R_1 - R_2$ contains all ordered pairs that are in the relation R_1 that do not occur in the relation R_2 .

$$\begin{aligned}R_1 - R_2 &= \{(a, b) \in \mathbf{R}^2 \mid a > b \text{ and } a \not< b\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a > b \text{ and } a < b\} \\ &= \emptyset\end{aligned}$$

Since $a > b$ and $a < b$ can never be true at the same time, $R_1 - R_2$ cannot contain any ordered pairs.

(f) $R_2 - R_1$ contains all ordered pairs that are in the relation R_2 that do not occur in the relation R_1 .

$$\begin{aligned}
 R_2 - R_1 &= \{(a, b) \in \mathbf{R}^2 \mid a \geq b \text{ and } a \not\leq b\} \\
 &= \{(a, b) \in \mathbf{R}^2 \mid a \geq b \text{ and } a \leq b\} \\
 &= \{(a, b) \in \mathbf{R}^2 \mid a = b\} \\
 &= R_5
 \end{aligned}$$

(g) $R_1 \oplus R_3$ contains all ordered pairs that are in the relation R_1 or are in the relation R_3 , but not in both.

$$\begin{aligned}
 R_1 \oplus R_3 &= \{(a, b) \in \mathbf{R}^2 \mid (a > b \text{ or } a < b) \text{ and not } (a > b \text{ and } a < b)\} \\
 &= \{(a, b) \in \mathbf{R}^2 \mid a \neq b \text{ and not } \mathbf{F}\} \\
 &= \{(a, b) \in \mathbf{R}^2 \mid a \neq b \text{ and } \neg \mathbf{F}\} \\
 &= \{(a, b) \in \mathbf{R}^2 \mid a \neq b \text{ and } \mathbf{T}\} && \neg \mathbf{F} \equiv \mathbf{T} \\
 &= \{(a, b) \in \mathbf{R}^2 \mid a \neq b\} && p \wedge \mathbf{T} \equiv p \\
 &= R_6
 \end{aligned}$$

Note that the statement $a > b$ and $a < b$ is always false \mathbf{F} for all pairs of real numbers.

(h) $R_2 \oplus R_4$ contains all ordered pairs that are in the relation R_2 or are in the relation R_4 , but not in both.

$$\begin{aligned} R_2 \oplus R_4 &= \{(a, b) \in \mathbf{R}^2 \mid (a \geq b \text{ or } a \leq b) \text{ and not } (a \geq b \text{ and } a \leq b)\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid \mathbf{T} \text{ and not } (a = b)\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid \mathbf{T} \text{ and } \neg(a = b)\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid \mathbf{T} \text{ and } a \neq b\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a \neq b\} && p \wedge \mathbf{T} \equiv p \\ &= R_6 \end{aligned}$$

Note that the statement $a \geq b$ or $a \leq b$ is always true \mathbf{T} for all pairs of real numbers.

SOLUTION

$$R_1 = \{(a, b) \in \mathbf{R}^2 \mid a > b\}$$

$$R_2 = \{(a, b) \in \mathbf{R}^2 \mid a \geq b\}$$

$$R_3 = \{(a, b) \in \mathbf{R}^2 \mid a < b\}$$

$$R_4 = \{(a, b) \in \mathbf{R}^2 \mid a \leq b\}$$

$$R_5 = \{(a, b) \in \mathbf{R}^2 \mid a = b\}$$

$$R_6 = \{(a, b) \in \mathbf{R}^2 \mid a \neq b\}$$

(a) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned} R_1 \circ R_1 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : (a, b) \in R_1 \text{ and } (b, c) \in R_1\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a > b \text{ and } b > c\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a > c\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid a > c\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a > b\} \\ &= R_1 \end{aligned}$$

$a > c$ is independent of b

Let us rename c as b

(b) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned} R_1 \circ R_2 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : (a, b) \in R_2 \text{ and } (b, c) \in R_1\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a \geq b \text{ and } b > c\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a > c\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid a > c\} && a > c \text{ is independent of } b \\ &= \{(a, b) \in \mathbf{R}^2 \mid a > b\} && \text{Let us rename } c \text{ as } b \\ &= R_1 \end{aligned}$$

(c) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned} R_1 \circ R_3 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : (a, b) \in R_3 \text{ and } (b, c) \in R_1\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a < b \text{ and } b > c\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid \mathbf{T}\} && \exists b \in \mathbf{R} : a < b \text{ and } b > c \text{ is always true} \\ &= \{(a, c) \in \mathbf{R}^2\} \\ &= \mathbf{R}^2 \end{aligned}$$

$\exists b \in \mathbf{R} : a < b$ and $b > c$ is always true, because we can always chose $b = \max(a, c) + 1$.

(d) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned}
 R_1 \circ R_4 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : (a, b) \in R_4 \text{ and } (b, c) \in R_1\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a \leq b \text{ and } b > c\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \mathbf{T}\} && \exists b \in \mathbf{R} : a \leq b \text{ and } b > c \text{ is always true} \\
 &= \{(a, c) \in \mathbf{R}^2\} \\
 &= \mathbf{R}^2
 \end{aligned}$$

$\exists b \in \mathbf{R} : a \leq b$ and $b > c$ is always true, because we can always chose $b = \max(a, c) + 1$.

(e) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned}
 R_1 \circ R_5 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : (a, b) \in R_5 \text{ and } (b, c) \in R_1\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a = b \text{ and } b > c\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a > c\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid a > c\} && a > c \text{ is independent of } b \\
 &= \{(a, b) \in \mathbf{R}^2 \mid a > b\} && \text{Let us rename } c \text{ as } b \\
 &= R_1
 \end{aligned}$$

(f) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned}R_1 \circ R_6 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : (a, b) \in R_6 \text{ and } (b, c) \in R_1\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a \neq b \text{ and } b > c\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid \mathbf{T}\} && \exists b \in \mathbf{R} : a \neq b \text{ and } b > c \text{ is always true} \\ &= \{(a, c) \in \mathbf{R}^2\} \\ &= \mathbf{R}^2\end{aligned}$$

$\exists b \in \mathbf{R} : a \neq b$ and $b > c$ is always true, because we can chose $b = c + 1$ when $c + 1 \neq a$ and $b = c + 2$ when $c + 1 = a$.

(g) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned}R_2 \circ R_3 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : (a, b) \in R_3 \text{ and } (b, c) \in R_2\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a < b \text{ and } b \geq c\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid \mathbf{T}\} && \exists b \in \mathbf{R} : a < b \text{ and } b \geq c \text{ is always true} \\ &= \{(a, c) \in \mathbf{R}^2\} \\ &= \mathbf{R}^2\end{aligned}$$

$\exists b \in \mathbf{R} : a < b$ and $b \geq c$ is always true, because we can chose $b = \max(a, c) + 1$.

(h) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned}R_3 \circ R_3 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : (a, b) \in R_3 \text{ and } (b, c) \in R_3\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a < b \text{ and } b < c\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a < c\} \\ &= \{(a, c) \in \mathbf{R}^2 \mid a < c\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a < b\} \\ &= R_3\end{aligned}$$

$a < c$ is independent of b

Let us rename c as b

Ex 6

Let R_1 and R_2 be the “divides” and “is a multiple of” relations on the set of all positive integers, respectively. That is, $R_1 = \{(a, b) \mid a \text{ divides } b\}$ and $R_2 = \{(a, b) \mid a \text{ is a multiple of } b\}$. Find

- a) $R_1 \cup R_2$.
- b) $R_1 \cap R_2$.
- c) $R_1 - R_2$.
- d) $R_2 - R_1$.
- e) $R_1 \oplus R_2$.

DEFINITIONS

The **composite** $S \circ R$ consists of all ordered pairs (a, c) for which there exists an element b such that $(a, b) \in R$ and $(b, c) \in S$

SOLUTION

$$R_1 = \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b\}$$

$$R_2 = \{(a, b) \in \mathbf{R}^2 \mid a \text{ is a multiple of } b\}$$

(a) The union of two relations contains all ordered pairs that are in either relation.

$$\begin{aligned} R_1 \cup R_2 &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ or } a \text{ is a multiple of } b\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ or } b \text{ divides } a\} \end{aligned}$$

Note: a is a multiple of b if and only if b divides a .

(b) The intersection of two relations contains all ordered pairs that are in both relations.

$$\begin{aligned} R_1 \cap R_2 &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ and } a \text{ is a multiple of } b\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ and } b \text{ divides } a\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a = \pm b \text{ and } a \neq 0\} \end{aligned}$$

Note: You cannot divide by 0.

(c) $R_1 - R_2$ contains all ordered pairs that are in the relation R_1 that do not occur in the relation R_2 .

$$\begin{aligned}R_1 - R_2 &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ and } a \text{ is not a multiple of } b\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ and } b \text{ does not divide } a\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ and } a \neq \pm b\}\end{aligned}$$

(d) $R_2 - R_1$ contains all ordered pairs that are in the relation R_2 that do not occur in the relation R_1 .

$$\begin{aligned}R_2 - R_1 &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ is a multiple of } b \text{ and } a \text{ does not divide } b\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid b \text{ divides } a \text{ and } a \text{ does not divide } b\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid b \text{ divides } a \text{ and } a \neq \pm b\}\end{aligned}$$

(e) $R_1 \oplus R_2$ contains all ordered pairs that are in the relation R_1 or R_2 , but not in both.

$$\begin{aligned} R_1 \oplus R_2 &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ or } b \text{ is a multiple of } a, \text{ but not both}\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid (a \text{ divides } b \text{ or } b \text{ divides } a) \text{ and } a \neq \pm b\} \end{aligned}$$